

Normal domains with monomial presentations

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Abstract

Let A be a finitely generated commutative algebra over a field K with a presentation $A = K\langle X_1, \dots, X_n \mid R \rangle$, where R is a set of monomial relations in the generators X_1, \dots, X_n . So $A = K[S]$, the semigroup algebra of the monoid $S = \langle X_1, \dots, X_n \mid R \rangle$. We characterize, purely in terms of the defining relations, when A is an integrally closed domain, provided R contains at most two relations. Also the class group of such algebras A is calculated.

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1 Introduction

Normal Noetherian domains, also called integrally closed Noetherian domains, are of fundamental importance in several areas of mathematics. In the literature one can find several concrete constructions of such rings that are algebras over a field K and that have a presentation in which the relations are of monomial type. Such algebras are commutative semigroup algebras $K[S]$ of a finitely generated abelian and cancellative monoid S (that is, S is a submonoid of a finitely generated abelian group G). Within the context of commutative ring theory, these algebras received a lot of attention (see for example [2, 9]). We recall some well known facts. First, a commutative semigroup algebra $K[S]$ of a monoid S is Noetherian if and only if S is finitely generated. In this case $K[S]$ also is finitely presented. Second, $K[S]$ is a domain if and only if S is a submonoid of a torsion free abelian group. Recall that an affine semigroup S is a finitely generated submonoid of a free abelian group. If, moreover, the unit groups $U(S)$ is trivial, that is $U(S) = \{1\}$, then S is said to be positive. Third (see [2, Proposition 6.1.4] or [17, Proposition 13.5]), if M is an affine monoid then $K[M]$ is normal if and only if M is normal (i.e. if $g \in MM^{-1}$, the group of fractions of M , and $g^n \in M$ for some $n \geq 1$ then $g \in M$). Moreover, such monoids M are precisely the monoids of the form $U \times M'$, where U is a finitely generated free abelian group and M' is a positive monoid so that $M' = (M')(M')^{-1} \cap F^+$, with F^+ the positive cone of a free abelian group F . Note that if M is positive and of rank n , that is MM^{-1} is a group of

torsion free rank n , then M is isomorphic to a submonoid of \mathbb{N}^n , a free abelian monoid of rank n . So, normality of $K[S]$ is a homogeneous property, i.e., a condition on the monoid S . This was one of the motivating reasons for these investigations. Furthermore, it is well known that $\text{cl}(K[S])$, the class group of $K[S]$, is naturally isomorphic with $\text{cl}(S)$, the class group of S (see for example [1, Theorem 2.3.1]). As an application one obtains much easier calculations for the class group of several classical examples of Noetherian normal domains. So the study of normal positive monoids is relevant in the context of number theory. Another reason for their importance is their connection to geometry, especially in the context of toric varieties and convex polytopes (see for example [1, 13, 17, 18] for an extensive bibliography of the subject, its computational aspects and applications to other fields).

The study of the above problems is also crucial in a noncommutative setting. Indeed, noncommutative maximal orders of the form $K[S]$, with S a cancellative nonabelian monoid, appear in the search of set-theoretical solutions of the quantum Yang-Baxter equation. Gateva-Ivanova and Van den Bergh [8] and Etingof, Schedler and Soloviev in [6] showed that such solutions are determined by monoids M of I -type. In [10] this was extended to the larger class of monoids of IG -type. Such monoids are contained in a finitely generated abelian-by-finite group and their algebras share many properties with commutative polynomial algebras. In particular, they are maximal orders in a division algebra and the algebraic structure of M is determined by a normal positive submonoid and a finite solvable group acting on it. More generally, as shown in [11], every prime maximal order $K[S]$ satisfying a polynomial identity is in some sense built on the basis of a normal abelian submonoid of S and every abelian normal monoid can be used to construct a family of noncommutative maximal orders. For more details on noncommutative orders we refer the reader to [12].

In this paper we deal with Noetherian commutative semigroup algebras $K[S]$ that are defined by at most two monomial relations. We obtain a characterization purely in terms of the defining relations, of when such an algebra is a normal domain. It is easily seen that if $K[S]$ is such an algebra then S has codimension at most 2. Recall that S has codimension $n - d$ if it is generated by n elements and $S \subseteq \mathbb{N}^d$. Recently Dueck, Hoşten and Sturmfels obtained necessary conditions for such algebras to be normal. In order to state this we recall that given a term order \prec on the free abelian monoid $F = \langle u_1, \dots, u_n \rangle$, the initial ideal I_\prec of S (corresponding to this order) is the ideal of F consisting of all leading (highest) monomials in every relation that holds in S .

Proposition 1.1 ([5, Theorem 1]) *Suppose S is a positive monoid of codimension two. If S is normal then S has a square free initial ideal (that is, a semiprime ideal in S).*

If, moreover, S is a homogeneous monoid (that is, S is defined by relations that are homogeneous with respect to the total degree) then the converse follows from Proposition 13.15 in [17]. The latter says that if S is a homogeneous submonoid of \mathbb{Z}^d such that for some order \prec the corresponding initial ideal I_\prec

is square free, then S is a normal monoid. Theorem 2 in [5] also says that if S is a positive monoid of codimension $n - d$ then there is an algorithm to decide whether A is normal, whose running time is polynomial.

From the characterization proved in this paper it follows that the converse of Proposition 1.1 holds for an arbitrary positive monoid S defined by at most two relations (so without the homogeneous assumption). Exercise 13.17 in [17] implies that this converse is false in general. It is worth mentioning that other constraints for normality of abelian monoids have been studied in [14, 15, 16].

As an application, we determine the class group $\text{cl}(S)$ in terms of the combinatorial data contained in the defining relations.

2 One-relator monoids

Our main aim is to describe when a positive monoid defined by at most two relations is normal. A first important obstacle to overcome is to determine when such monoids are cancellative, i.e., when they are contained in a group and next to decide when this group can be assumed torsion free. Because of the comments given in the introduction, and since we are mainly interested in such monoids that are normal, we only need to deal with monoids S so that $U(S) = \{1\}$. In this context we mention that in [3] an algorithm of Contejean and Devie is used to determine whether a finitely generated monoid given by a presentation is cancellative.

We will use the following notation. By FaM_n we denote a free abelian monoid of rank n . If $\text{FaM}_n = \langle u_1, \dots, u_n \rangle$ and $w = u_1^{a_1} \cdots u_n^{a_n} \in \text{FaM}_n$, then put $\text{supp}(w) = \{u_i \mid a_i \neq 0\}$, the support of w , and $\text{Hsupp}(w) = \{u_j \mid a_j > 1\}$. We say that w is square free if $\text{Hsupp}(w) = \emptyset$. Now, suppose S has a presentation

$$S = \langle u_1, \dots, u_n \mid w_1 = v_1, \dots, w_m = v_m \rangle,$$

where w_i, v_i are nonempty words in the free abelian monoid $\text{FaM}_n = \langle u_1, \dots, u_n \rangle$. Clearly, $U(S) = \{1\}$ and if S is cancellative, then we may assume it has a presentation with

$$\text{supp}(w_i) \cap \text{supp}(v_i) = \emptyset,$$

for all i .

Recall from Lemma 6.1 in [14] that if $K[S]$ is a normal domain and $w_i = v_i$ is independent of the other defining relations then at least one of w_i or v_i is square free.

Proposition 2.1 *Let S be an abelian monoid defined by the presentation*

$$\langle u_1, \dots, u_n \mid u_1 \cdots u_k = u_{k+1}^{a_{k+1}} \cdots u_n^{a_n} \rangle$$

for some positive integers a_{k+1}, \dots, a_n and some $k < n$. Let $\text{FaM}_{k(n-k)} = \langle x_{i,j} \mid 1 \leq i \leq k, 1 \leq j \leq n-k \rangle$, a free abelian monoid of rank $k(n-k)$. For $1 \leq j \leq k$ put

$$v_j = x_{j,1}^{a_{k+1}} x_{j,2}^{a_{k+2}} \cdots x_{j,n-k}^{a_n},$$

and for $k+1 \leq j \leq n$ put

$$v_j = x_{1,j-k} x_{2,j-k} \cdots x_{k,j-k}.$$

Then $S \cong V = \langle v_1, \dots, v_n \rangle \subseteq \text{FaM}_{k(n-k)}$ (in particular, S is cancellative) and $\langle v_1, \dots, v_n \rangle$ has $v_1 \cdots v_k = v_{k+1}^{a_{k+1}} \cdots v_n^{a_n}$ as its only defining relation.

Proof. Let $V = \langle v_1, \dots, v_n \rangle \subseteq \text{FaM}_{k(n-k)}$. Clearly, $v_1 \cdots v_k = v_{k+1}^{a_{k+1}} \cdots v_n^{a_n}$ and thus $V = \langle v_1, \dots, v_n \rangle$ is a natural homomorphic image of S .

Since all $a_i \neq 0$, it is easy to see that every relation in V (with disjoint supports with respect to the v_i 's) must involve all generators v_i . Moreover, since v_1, v_{k+1} are the only generators involving $x_{1,1}$, it follows that in such a relation v_1, v_{k+1} are on opposite sides of the equality. And also v_{k+2}, \dots, v_n must be on the side opposite to v_1 (look at the appearance of $x_{1,2}, x_{1,3}, \dots, x_{1,n-k}$ in order to see this). Similarly, by looking at the appearance of $x_{2,1}, x_{3,1}, \dots, x_{k,1}$, we get that v_2, \dots, v_k must be on the side opposite to v_{k+1} . It follows that every relation in V , possibly after cancellation, must be of the form

$$v_1^{c_1} \cdots v_k^{c_k} = v_{k+1}^{c_{k+1}} \cdots v_n^{c_n} \quad (1)$$

for some positive integers c_j . Again, using the fact that $x_{i,j}$'s are independent and comparing the exponent of $x_{i,j}$ on both sides of (1), we get that $a_{k+j}c_i = c_{k+j}$ for $1 \leq i \leq k$ and $j = 1, 2, \dots, n-k$. This implies that $c_1 = c_2 = \cdots = c_k$. Hence relation (1) is of the form $(v_1 \cdots v_k)^{c_1} = (v_{k+1}^{a_{k+1}} \cdots v_n^{a_n})^{c_1}$. So it is a consequence of the relation defining S with every u_j replaced by v_j . It follows that $V \cong S$. ■

Note that one can verify that the monoid V , as described in the previous proposition, is such that $V = VV^{-1} \cap \text{FaM}_{k(n-k)}$. So, by the comments given in the introduction, V is normal. Alternatively, it easily follows from the defining relation that $S = \bigcup_{1 \leq i \leq k} F_i$, with $F_i = \langle u_j \mid 1 \leq j \leq n, j \neq i \rangle$ a free abelian monoid with group of quotients SS^{-1} . Since each F_i is normal we thus obtain that S is normal as well ([1, Proposition 3.1.1]).

Hence, the Proposition 2.1 and its preceding comment yield at once a description one-relator positive monoids that are normal.

Proposition 2.2 *Let S be the abelian monoid defined by the presentation*

$$\langle u_1, \dots, u_n \mid w_1 = w_2 \rangle,$$

with nonempty words $w_1 = u_1^{a_1} \cdots u_k^{a_k}$, $w_2 = u_{k+1}^{a_{k+1}} \cdots u_n^{a_n}$, where $k < n$, and each a_i is a nonnegative integer. The following conditions are equivalent.

1. *The semigroup S is a normal positive monoid, normal (or equivalently, the semigroup algebra $K[S]$ is a normal domain).*
2. *$\text{Hsupp}(w_1) = \emptyset$ or $\text{Hsupp}(w_2) = \emptyset$.*

In the remainder of this section we describe the class group $\text{cl}(S)$ of a one-relator normal positive monoid S . For convenience sake we recall some terminology for an affine normal monoid M (see [4, 9]; at an algebra level we refer to [7]). For a subset I of MM^{-1} we put $(M : I) = \{g \in MM^{-1} \mid gI \subseteq M\}$. A fractional ideal I of M is a subset of MM^{-1} so that $MI \subseteq I$ and $mI \subseteq M$ for some $m \in M$. A fractional ideal is said to be divisorial if $I = I^*$, where $I^* = (M : (M : I))$. The set of all divisorial fractional ideals is denoted by $D(M)$. It is a free abelian group for the divisorial product $I * J = (IJ)^*$, for $I, J \in D(M)$, with basis the set of minimal prime ideals. Also, $M = \bigcap M_P$, where the intersection runs over all minimal primes of M , and all localizations M_P are discrete valuation monoids (see for example [4, 9]). Furthermore, for an ideal I of M one has, in the divisorial group $D(M)$, that $I = (\prod_P P^{n(P)})^*$ if and only if $M_P I = M_P P^{n(P)}$, with all $n_P \geq 0$. Moreover, $n_P > 0$ if and only if $I \subseteq P$.

By definition $\text{cl}(M) = D(M)/P(M)$, where $P(M) = \{Mg \mid g \in MM^{-1}\}$.

Let S be again as in Proposition 2.2. We will use the same notation for the generators u_i of the free monoid FaM_n and for their images in S , if unambiguous. So, every S_{u_j} in $D(S)$ is a (unique) product of the minimal primes of S . In the following lemma we compute these decompositions provided all a_i are positive integers. Clearly, in this case, the minimal primes of S are the ideals P_{yz} generated by the set $\{u_y, u_z\}$, where $y \in \{1, \dots, k\}$, $z \in \{k+1, \dots, n\}$.

Lemma 2.3 *Let $S = \langle u_1, \dots, u_n \mid u_1 \cdots u_k = u_{k+1}^{a_{k+1}} \cdots u_n^{a_n} \rangle$ be a normal monoid, with all $a_i \geq 1$, and let P_{yz} denote the minimal prime ideal of S that is generated by the set $\{u_y, u_z\}$, where $y \in \{1, \dots, k\}$, $z \in \{k+1, \dots, n\}$. Then*

$$Su_z = P_{1z} * \cdots * P_{kz} \quad \text{and} \quad Su_y = P_{yk+1}^{a_{k+1}} * \cdots * P_{yn}^{a_n},$$

for $z \in \{k+1, \dots, n\}$ and $y \in \{1, \dots, k\}$.

Proof. First, let $y \in \{1, \dots, k\}$. Note that the only minimal primes containing u_y are $P_{y,z}$, with $z \in \{k+1, \dots, n\}$. Hence $Su_y = \left(\prod_{k \leq z \leq n} P_{y,z}^{e(z)} \right)^*$, with $e(z) \geq 1$. Furthermore, in the localization $T = S_{P_{y,z}}$ we have that u_i, u_j are invertible for $y \neq i \in \{1, \dots, k\}$ and $z \neq j \in \{k+1, \dots, n\}$. Hence, from the defining relation it follows that $Tu_y = Tu_z^{a_z}$ and thus also $Tu_y = TP_{y,z}^{a_z}$. Consequently, $e(z) = a_z$ and thus $Su_y = P_{yk+1}^{a_{k+1}} * \cdots * P_{yn}^{a_n}$, as desired.

Second, assume $z \in \{k+1, \dots, n\}$. Then, for any $y \in \{1, \dots, k\}$, it is easily seen from the defining relation that $Tu_y \subseteq Tu_z$, with $T = S_{P_{y,z}}$. Thus $Tu_z = TP_{y,z}$. Therefore, as above, $Su_z = P_{1z} * \cdots * P_{kz}$. ■

Theorem 2.4 *Let $S = \langle u_1, \dots, u_n, \dots, u_m \mid u_1 \cdots u_k = u_{k+1}^{a_{k+1}} \cdots u_n^{a_n} \rangle$ be a positive normal monoid (with all $a_i \geq 1$ and $n \leq m$). Then*

$$\text{cl}(K[S]) \cong \text{cl}(S) \cong \mathbb{Z}^{k(n-k)-(n-1)} \times (\mathbb{Z}_d)^{k-1},$$

where $d = \gcd(a_{k+1}, \dots, a_n)$, $k(n-k)$ is the number of minimal primes in S not containing one of the independent generators u_{n+1}, \dots, u_m , and $m-1$ is the torsion free rank of SS^{-1} .

Proof. Clearly, $S = S' \times \text{FaM}_{m-n}$, where $\text{FaM}_r = \langle a_{n+1}, \dots, a_m \rangle$ is a free abelian monoid, and $S' = \langle u_1, \dots, u_n \mid u_1 \cdots u_k = u_{k+1}^{a_{k+1}} \cdots u_n^{a_n} \rangle$. So, S is normal if and only if S' is normal. Because also $\text{cl}(S') = \text{cl}(S)$, we may assume $S = S'$.

Clearly, the result is true for $k = 1$. So assume that $k \geq 2$. As there are $k(n-k)$ minimal primes P_{yz} in S (with $1 \leq y \leq k, k+1 \leq z \leq n$), we get that $D(S) \cong \mathbb{Z}^{k(n-k)}$. On the other hand, $P(S) = \text{gr}(Su_i \mid i = 1, \dots, n)$. By Lemma 2.3, $Su_j = \left(\prod_{l=1}^k P_{lj}\right)^*$, $Su_i = \left(\prod_{l=k+1}^n P_{il}^{a_l}\right)^*$ for $i \in \{1, \dots, k\}, j \in \{k+1, \dots, n\}$. We consider

$$\text{cl}(S) = \text{gr}(P_{yz} \mid y \in \{1, \dots, k\}, z \in \{k+1, \dots, n\}) / \text{gr}(Su_i \mid 1 \leq i \leq n).$$

as a finitely generated \mathbb{Z} -module. So its presentation corresponds to an integer matrix M of size $k(n-k) \times n$. The rows of M are indexed by elements of the set $R = \{(i, j) \mid i \in \{1, \dots, k\}, j \in \{k+1, \dots, n\}\}$. We agree on the lexicographic ordering of the set of rows of M . The columns are indexed by $C = \{1, 2, \dots, n\}$, where the i -th column corresponds to the generator Su_i , written as a vector in terms of the minimal primes of S .

We consider the block decomposition of M determined by the following partitions of the sets C and R of columns and rows: $C = D_1 \cup D_2$, where $D_1 = \{1, \dots, k\}$ and $D_2 = \{k+1, \dots, n\}$ and $R = R_1 \cup \dots \cup R_k$, where $R_i = \{(i, j) \mid j = k+1, \dots, n\}$. Then M has the following form:

$$\begin{pmatrix} a_{k+1} & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ a_{k+2} & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ 0 & a_{k+1} & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & a_{k+2} & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & a_n & 0 & \cdots & 0 & 0 & \cdots & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & a_{k+1} & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & a_{k+2} & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & a_n & 0 & 0 & \cdots & 1 \end{pmatrix}$$

We subtract the subsequent rows of the last row block R_k from the corresponding rows of all other row blocks. Then from column k we subtract $\sum_{i=k+1}^n a_i C_i$, where C_i denotes the i -th column. The obtained matrix M' has the (R_k, C) -block of the form $M'_{R_k, C} = (0, I)$, where I is the $(n-k) \times (n-k)$ identity matrix and $M'_{R_i D_2}$ is a zero matrix for every $i \neq k$. Let $T = R \setminus R_k$. The last column of the submatrix M'_{T, D_1} has the form $(-a_{k+1}, \dots, -a_n, \dots, -a_{k+1}, \dots, -a_n)^t$,

hence adding all other columns of M'_{T,D_1} to it, we get a matrix N such that $N = M_{T,D_1}$. Clearly, the normal form of N involves $k - 1$ entries equal to $d = \gcd(a_{k+1}, \dots, a_n)$ and no other nonzero entries. The result follows. ■

3 Two-relator monoids

In this section we obtain a characterization of normal positive monoids that are defined by two relations. The class group of such monoids S , and therefore of the corresponding algebras $K[S]$, is also determined.

Theorem 3.1 *Let $S = \langle u_1, \dots, u_n \rangle$ be a finitely presented abelian monoid with independent defining relations $w_1 = w_2$ and $w_3 = w_4$ and, $|\text{supp}(w_i)| \geq 1$ for all i . The following conditions are equivalent.*

1. *The semigroup S is a normal positive monoid (or equivalently, the semigroup algebra $K[S]$ is a normal domain).*
2. *S is a positive monoid with an initial ideal $I_{<}$ of S that is square free.*
3. *The following conditions hold:*

- (a) $\text{supp}(w_1) \cap \text{supp}(w_2) = \emptyset, \text{supp}(w_3) \cap \text{supp}(w_4) = \emptyset,$
- (b) $\text{Hsupp}(w_1) = \emptyset$ or $\text{Hsupp}(w_2) = \emptyset,$
- (c) $\text{Hsupp}(w_3) = \emptyset$ or $\text{Hsupp}(w_4) = \emptyset,$
- (d) *if there exist $i \in \{1, 2\}, j \in \{3, 4\}$ such that $\text{supp}(w_i) \cap \text{supp}(w_j) \neq \emptyset$, then one of the following properties holds (we may assume for simplicity that $i = 1$ and $j = 3$):*
 - $\text{supp}(w_k) \cap \text{supp}(w_l) = \emptyset$ for all pairs $\{k, l\} \neq \{1, 3\}$ with $k \neq l$, and $\text{Hsupp}(w_2) = \emptyset$ or $\text{Hsupp}(w_4) = \emptyset,$
 - *there exists a pair $k \neq l$ such that $\{2, 4\} \neq \{k, l\} \neq \{1, 3\}$ and $\text{supp}(w_k) \cap \text{supp}(w_l) \neq \emptyset$ (for simplicity assume $k = 2, l = 3$), $\text{supp}(w_4) \cap \text{supp}(w_i) = \emptyset$ for $i = 1, 2, 3$ and $\text{Hsupp}(w_4) = \emptyset.$*

Proof. Note that $S = S_1 \times S_2$, where S_2 is the free abelian monoid generated by

$$\{u_1, \dots, u_n\} \setminus \left(\bigcup_{i=1}^4 \text{supp}(w_i) \right)$$

and

$$S_1 = \langle \bigcup_{i=1}^4 \text{supp}(w_i) \rangle.$$

Since S_2 is a normal positive monoid, it follows that S is a normal positive monoid if and only if S_1 is such a monoid, i.e. we may assume that $\{u_1, \dots, u_n\} = \bigcup_{i=1}^4 \text{supp}(w_i)$.

It follows from Proposition 1.1 that (1) implies (2). We now prove (2) implies (3). So assume that $I = I_{\prec}$ is a square free ideal for some term order \prec and S is a positive monoid. In order to prove (3.a) suppose for example that $\text{supp}(w_1) \cap \text{supp}(w_2) \neq \emptyset$. Then write $w_1 = uw'_1, w_2 = uw'_2$ for a nontrivial word u and some w'_1, w'_2 such that $\text{supp}(w'_1) \cap \text{supp}(w'_2)' = \emptyset$. Hence, in S , we have $w'_1 = w'_2$, and thus each of w'_1, w'_2 is divisible by some of the w_j 's. So, by symmetry, we may assume that $w'_1 = w_3z$ and $w'_2 = w_4y$. Let m be the maximal positive integer such that $w'_1 = w_3^m z'$ and $w'_2 = w_4^m y'$ for some z', y' . Then $z' = y'$ holds in S and it follows that $z' = y'$ as words (otherwise y', z' would be again divisible by w_3, w_4 , respectively, contradicting the choice of m). It follows that the relation $w_1 = w_2$ is a consequence of $w_3 = w_4$, a contradiction. So (3.a) follows.

In order to prove conditions (3.b), (3.c) and (3.d) we introduce the following notation. For a word w in u_1, \dots, u_n we define $\sqrt{w} = x_1 \cdots x_p$ where $\text{supp}(w) = \{x_1, \dots, x_p\}$.

Note that if $\text{supp}(w_1) \cap \text{supp}(w_3) \neq \emptyset$ then we must have that $\text{supp}(w_2) \cap \text{supp}(w_4) = \emptyset$. Indeed, for otherwise, the ideal $K[S](w_1 - w_2, w_3 - w_4) \subseteq K[S](u_i, u_j)$, for some $i \neq j$. Since both ideals are height two primes, they must be equal, a contradiction (note that S is, by assumption, a positive monoid and thus $K[S]$ is a domain). If $\text{supp}(w_2) \cap \text{supp}(w_3) \neq \emptyset$ then, by the same reasoning, $\text{supp}(w_1) \cap \text{supp}(w_4) = \emptyset$. Hence we have shown that either all $\text{supp}(w_i)$ are disjoint or $\text{supp}(w_i) \cap \text{supp}(w_j) \neq \emptyset$ for exactly one pair i, j or this holds for exactly two pairs and these pairs are of the form i, j and i, m for some i, j, m . So, by symmetry, it is enough to deal with the three cases considered below.

If all $\text{supp}(w_i)$ are disjoint then let for example $w_2 \prec w_1$ and $w_4 \prec w_3$. It easily follows from the assumption that w_1, w_3 must be square free and hence (3.b), (3.c) and (3.d) hold.

Next, assume that $\text{supp}(w_1) \cap \text{supp}(w_3) \neq \emptyset$ and $\text{supp}(w_i) \cap \text{supp}(w_j) = \emptyset$ for every pair $(i, j) \neq (1, 3)$.

To prove (3.d) we need to show that $\text{Hsupp}(w_i) = \emptyset$ for $i = 2$ or $i = 4$. So, suppose otherwise, that is, w_2, w_4 are not square free. Then $w_1, w_3 \in I$ and $w_2 \prec w_1, w_4 \prec w_3$ (because for example if $w_1 \prec w_2$ then $w_2 \in I$, so $w_2 \neq \sqrt{w_2} \in I$, whence $\sqrt{w_2}$ is in a nontrivial relation in S , but it cannot be divisible by any of the words $w_i, i = 1, 2, 3, 4$, a contradiction). Let $w_k = ww'_k$ for $k = 1, 3$, where $\text{supp}(w'_1) \cap \text{supp}(w'_3) = \emptyset$. Then $w_3w'_1 = w_1w'_3$ as words and, in S , we have $w_3w'_1 = w_4w'_1$ and $w_1w'_3 = w_2w'_3$. So one of the words $w_4w'_1, w_2w'_3$ is in I . Therefore $\sqrt{w_4}w'_1 \in I$ or $\sqrt{w_2}w'_3 \in I$. Say, for example, that the former holds. Then $\sqrt{w_4}w'_1$ is in a nontrivial relation in S . But it is easy to see that $\sqrt{w_4}w'_1$ cannot have w_i as a subword for every $i = 1, 2, 3, 4$. This contradiction establishes assertion (3.d).

To prove (3.b) and (3.c) in this case, suppose for example that $\text{Hsupp}(w_2) = \emptyset$ and $\text{Hsupp}(w_3) \neq \emptyset \neq \text{Hsupp}(w_4)$. An argument as before shows that $w_4 \prec w_3 \in I$ and $w_4 \notin I$. Hence $w_3 \neq \sqrt{w_3} \in I$. Then $\sqrt{w_3} = w_1x$ for a word x . The only relation in which w_1x can occur must be of the form $w_1x = w_2x$, whence we have $w_2 \prec w_1$. Write $w_3 = v_1v_3$ where $\text{supp}(v_1) = \text{supp}(w_1)$ and $\text{supp}(v_3) \cap \text{supp}(w_1) = \emptyset$. Let $k \geq 1$ be minimal such that w_3 divides $w_1^k v_3$.

Then $w_1^k v_3 = w_3 y$ for a subword y of w_1 such that $y \neq w_1$. So, in S , we get $w_2^k v_3 = w_1^k v_3 = w_3 y = w_4 y$. Since the word $w_4 y$ is not divisible by w_1, w_2, w_3 and $w_4 \neq \sqrt{w_4}$, it follows that $\sqrt{w_4 y} \notin I$, whence $w_2^k v_3 \in I$. Then $w_2 v_3 \in I$. But the only relation containing this word is $w_2 v_3 = w_1 v_3$. Since $w_2 v_3 \prec w_1 v_3$, we get a contradiction. We have shown that (3.b), (3.c) are satisfied.

Finally, consider the case where there are at exactly two overlaps between the supports of $w_i, i = 1, 2, 3, 4$. We may assume that $\text{supp}(w_1) \cap \text{supp}(w_3) \neq \emptyset$ and $\text{supp}(w_2) \cap \text{supp}(w_3) \neq \emptyset$. So $\text{supp}(w_4) \cap \text{supp}(w_j) = \emptyset$ for every $j \neq 4$.

Suppose that $\text{Hsupp}(w_4) \neq \emptyset$. Let $w_1 = ab$, $w_2 = cd$, $w_3 = a'c'e$, where $\text{supp}(a) = \text{supp}(a')$, $\text{supp}(c) = \text{supp}(c')$ and the remaining factors have pairwise disjoint supports. Let a_0, a'_0 be words of minimal length such that $aa_0 = a'a'_0$. Clearly, a'_0 is not divisible by a and $\text{supp}(a'_0) \cap \text{supp}(a_0) = \emptyset$.

Now $abc'e = cdc'e$ in S and $a'c'eb = w_4 b$ in S . So $cdc'ea_0 = w_4 ba'_0$ in S and hence one of these words is in I . If $w_4 ba'_0$ is in I then $\sqrt{w_4 ba'_0} \in I$, which is not possible because $\sqrt{w_4 ba'_0}$ cannot be rewritten in S (as $\sqrt{w_4}$ is a proper subword of w_4 with support independent of w_1, w_2, w_3 and ba'_0 is not divisible by any of w_1, w_2, w_3). Hence $cdc'ea_0 \in I$. Then $cdea_0 \in I$ because I is square free. But the only way to rewrite $cdea_0$ in S is $cdea_0 = abea_0$. Hence $abea_0 \prec cdea_0$, so also $w_1 = ab \prec cd = w_2$. However, repeating the above argument with the roles of w_1 and w_2 switched, we also get $w_2 \prec w_1$, a contradiction. We have proved that w_4 is square free, so (3.d) holds, and (3.c) also holds.

It remains to prove condition (3.b). Suppose that w_1, w_2 are not square free. By symmetry, we may assume that $w_2 \in I$. Then $\sqrt{w_1} \in I$ and in particular the word $\sqrt{w_1}$ it must be divisible by w_3 . But $\text{supp}(w_2) \cap \text{supp}(w_3) \neq \emptyset$ by the assumption, so $\text{supp}(w_2) \cap \text{supp}(w_1) \neq \emptyset$, a contradiction. This completes the proof of the fact that (3) is a consequence of (2).

Now we prove (3) implies (1). So, suppose that the four properties (3.a)-(3.d) hold. We claim that if S is embedded in a group then the group SS^{-1} is torsion free, and thus S is a positive affine semigroup. Note that in this case, SS^{-1} actually is a free abelian group of rank $n - 2$. Indeed, because of the assumptions there exists u_i and $\epsilon \in \{1, 2\}$ so that $u_i \in \text{supp}(w_\epsilon)$ and $\text{Hsupp}(w_\epsilon) = \emptyset$. Re-numbering the generators, if necessary, we may assume that $i = 1$. Then the relation $w_1 = w_2$ implies that $u_1 = wv^{-1}$ for some $w, v \in S$ with $\text{supp}(w) \cup \text{supp}(v) \cup \{u_1\} = \text{supp}(w_1) \cup \text{supp}(w_2)$, $u_1 \notin \text{supp}(w) \cup \text{supp}(v)$ and $\text{supp}(w) \cap \text{supp}(v) = \emptyset$. It follows that

$$SS^{-1} = \text{gr}(u_2, \dots, u_n \mid w_3(wv^{-1}, u_2, \dots, u_n) = w_4(wv^{-1}, u_2, \dots, u_n)).$$

If the second property of (3.d) holds then $\text{supp}(w_4) \cap (\bigcup_{i=1}^3 \text{supp}(w_i)) = \emptyset$ and $\text{Hsupp}(w_4) = \emptyset$. So, in particular, $u_1 \notin \text{supp}(w_4)$ and for $u_k \in \text{supp}(w_4)$ we have that $u_k \notin \text{supp}(w) \cup \text{supp}(v) \cup \text{supp}(w_3)$ and

$$u_k = w_3(wv^{-1}, u_2, \dots, u_n)u^{-1}$$

with $w_4 = uu_k$ and $\text{supp}(w_4) = \text{supp}(u) \cup \{u_k\}$. Hence we obtain that $SS^{-1} = \text{gr}(\{u_2, \dots, u_n\} \setminus \{u_k\})$ and this is a free abelian group of rank $n - 2$, as claimed. If, on the other hand, the first property of (3.d) holds then, without loss of

generality, we may assume that $\text{supp}(w_1) \cap \text{supp}(w_3) \neq \emptyset$, $\text{Hsupp}(w_2) = \emptyset$ and $u_1 \in \text{supp}(w_2)$. So, $u_1 \notin \text{supp}(w_3)$. If $\text{Hsupp}(w_3) = \emptyset$ then choose $u_k \in \text{supp}(w_3)$ and write $w_3 = u_k v'$ with $u_k \notin \text{supp}(v')$ and $\text{supp}(w_3) = \{u_k\} \cup \text{supp}(v')$. So $u_k = w_4(v')^{-1}$. Note that $u_1 \notin \text{supp}(w_4) \cup \text{supp}(v')$. It follows that $SS^{-1} = \text{gr}(\{u_2, \dots, u_n\} \setminus \{u_k\})$, a free abelian group of rank $n-2$. Finally, if $\text{Hsupp}(w_3) \neq \emptyset$ then $\text{Hsupp}(w_4) = \emptyset$. In this case write $w_4 = u_l v''$ for some v'' with $u_l \notin \text{supp}(v'')$ and $\text{supp}(w_4) = \{u_k\} \cup \text{supp}(v'')$. It follows that $SS^{-1} = \text{gr}(\{u_2, \dots, u_n\} \setminus \{u_l\})$, again a free abelian group of rank $n-2$, as desired.

So now we show that S is cancellative and thus embedded in Fa_{n-2} . By symmetry we can assume that $\text{Hsupp}(w_4) = \emptyset$. Then write

$$w_2 = y_1^{\gamma_1} \cdots y_q^{\gamma_q}, \quad w_4 = x_1 \cdots x_{p-1} x_p,$$

$\gamma_i \geq 1$, where $x_1, \dots, x_p, y_1, \dots, y_q \in \{u_1, \dots, u_n\}$, and $\text{supp}(w_4)$ does not intersect nontrivially the support of any other word in the defining relations.

Let F be the free abelian monoid with basis $\text{supp}(w_1) \cup \{y_1, \dots, y_q\} \cup \text{supp}(w_3) \cup \{x_1, \dots, x_{p-1}\}$. Then let $T = F/\rho$, where ρ is the congruence defined by the relation $w_1 = w_2$. Since $\text{Hsupp}(w_1) = \emptyset$ or $\text{Hsupp}(w_2) = \emptyset$, we know from Proposition 2.2 that T is a normal positive monoid. In particular, TT^{-1} is a torsion free group. Consider the semigroup morphism

$$f : T \times \langle u \rangle \longrightarrow TT^{-1}$$

defined by $f(t) = t$, for $t \in T$ and $f(u) = w_3 z^{-1}$ and $z = x_1 \cdots x_{p-1}$. Note that $f(w_3) = f(zu)$. Hence the above morphism induces the following natural morphisms

$$T \times \langle u \rangle \xrightarrow{\pi} (T \times \langle u \rangle) / \nu \xrightarrow{\bar{f}} TT^{-1},$$

with ν the congruence defined by the relation $w_3 = zu$. Put $M = (T \times \langle u \rangle) / \nu$ and note that

$$M \cong S.$$

For simplicity we denote $\pi(t)$ as \bar{t} , for $t \in T \times \langle u \rangle$. We note that $\pi|_T$, the restriction of π to T , is injective. Indeed, suppose $s, t \in T$ are such that $\pi(s) = \pi(t)$. Then

$$s - t \in K[T \times \langle u \rangle](zu - w_3),$$

an ideal in $K[T \times \langle u \rangle]$. So, $s - t = \alpha(zu - w_3)$, for some $\alpha \in K[T \times \langle u \rangle]$. Now $K[T \times \langle u \rangle]$ has a natural \mathbb{N} -gradation, with respect to the degree in u . Clearly, $s - t$ and w_3 have degree zero. Let α_h be the highest degree term of α with respect to this gradation. Then,

$$0 = \alpha_h zu.$$

Since $T \times \langle u \rangle$ is contained in a torsion free group, we know that $K[T \times \langle u \rangle]$ is a domain. So we get that $\alpha_h = 0$ and thus $\alpha = 0$. Hence $s = t$ and therefore indeed $\pi|_T$ is injective. So we will identify the element $\pi(t)$ with t , for $t \in T$.

Next we note that \bar{u} is a cancellable element in M . Indeed, let $\bar{x}, \bar{y} \in M$ and suppose $\bar{u} \bar{x} = \bar{u} \bar{y}$. This means that

$$ux - uy \in K[T \times \langle u \rangle](uz - w_3),$$

i.e.

$$ux - uy = \alpha(uz - w_3) \quad (2)$$

for some $\alpha \in K[T \times \langle u \rangle]$, where $x, y \in T \times \langle u \rangle$ are inverse images of \bar{x}, \bar{y} . Again consider the \mathbb{N} -gradation on $K[T \times \langle u \rangle]$ via the degree in u . Let α_0 be the zero degree component of α . Then it follows that

$$0 = \alpha_0 w_3.$$

Hence $\alpha_0 = 0$, as $K[T]$ is a domain, and thus

$$\alpha \in K[T \times \langle u \rangle]u.$$

Using again that $K[T \times \langle u \rangle]$ is a domain, we get from (2) that

$$x - y \in K[T \times \langle u \rangle](uz - w_3).$$

Hence $\bar{x} = \bar{y} \in M$, as desired.

In the above we thus have shown that \bar{u} is cancellable in M . Hence x_p is cancellable in S . The argument of the proof holds for all elements x_1, \dots, x_p . So, all elements x_1, \dots, x_p are cancellable in S . By a similar argument, if $\text{Hsupp}(w_2) = \emptyset$, this also holds for all elements $y_i \in \text{supp}(w_2) \setminus \text{supp}(w_3)$.

On the other hand, if $\text{Hsupp}(w_2) \neq \emptyset$ and thus $\text{Hsupp}(w_1) = \emptyset$, then similarly one shows that u_i is cancellable in S , for every $u_i \in \text{supp}(w_1) \setminus \text{supp}(w_3)$. Clearly, S is contained in its localization S_C , with respect to the multiplicatively closed set of the cancellable elements. In view of the form of the defining relations of S , this implies that S_C is a group. So S is a cancellative monoid in $SS^{-1} = \text{Fa}_{n-2}$.

Finally, we show that S is normal, by proving it is a union of finitely many finitely generated free abelian monoids. To so, note that conditions (3.a)-(3.d) imply that $\text{Hsupp}(w_i) = \emptyset$ and $\text{Hsupp}(w_j) = \emptyset$ for some $i \in \{1, 2\}$ and $j \in \{3, 4\}$. Furthermore, $\text{supp}(w_i) \cap \text{supp}(w_l) = \emptyset$ for all l with $l \neq i$, or $\text{supp}(w_j) \cap \text{supp}(w_l) = \emptyset$ for all l with $l \neq j$. Without loss of generality we may assume the former holds. Note that if $w_k = u_q$ for some k and some q then the assertion follows from Proposition 2.2. Hence, without loss of generality, we may assume that $|\text{supp}(w_k)| > 1$ for $k = 1, 2, 3, 4$.

Because $\text{Hsupp}(w_i) = \emptyset$, it is easily seen, using the relation involving w_i , that s can be written as a product of elements of $\{u_1, \dots, u_n\} \setminus \{u\}$ for some $u \in \text{supp}(w_i)$. If not all elements of $\text{supp}(w_j)$ occur in this product of s , then $s \in \langle \{u_1, \dots, u_n\} \setminus \{u, v\} \rangle$, with $v \in \text{supp}(w_j)$. Now because of the defining relations one easily sees that $\langle \{u_1, \dots, u_n\} \setminus \{u, v\} \rangle$ is a free abelian monoid, as desired. If, on the other hand, all elements of $\text{supp}(w_j)$ occur in the expression of s then, using the relation involving w_j (several times if needed) and using the

fact that $\text{supp}(w_i) \cap \text{supp}(w_l) = \emptyset$ for all $l \neq i$, we can reduce to the previous case. This ends the proof. ■

As a matter of example, it follows at once from Theorem 3.1 that the commutative algebra $K\langle u_1, u_2, u_3, u_4, u_5 \mid u_1 u_2 = u_3^2, u_1 u_3 = u_4 u_5 \rangle$ is a normal domain, while the commutative algebra $K\langle u_1, u_2, u_3, u_4 \mid u_1 u_2 = u_3^2, u_1 u_3 = u_4^2 \rangle$ is a domain that is not normal.

Finally, we describe the class group of positive monoid defined by two relations. We use the same notation as in the proof of Theorem 3.1. If $(\text{supp}(w_1) \cup \text{supp}(w_2)) \cap (\text{supp}(w_3) \cup \text{supp}(w_4)) = \emptyset$ then $S \cong S_1 \times S_2$, with $S_1 = \langle \text{supp}(w_1) \cup \text{supp}(w_2) \mid w_1 = w_2 \rangle$ and $S_2 = \langle \text{supp}(w_3) \cup \text{supp}(w_4) \mid w_3 = w_4 \rangle$. Clearly, in this case, $\text{cl}(S) \cong \text{cl}(S_1) \times \text{cl}(S_2)$, and the result follows from Theorem 2.4. So, assume S satisfies one of the properties in condition (3.d) in Theorem 3.1. Then, we can write

$$S = \langle u_1, \dots, u_n, \dots, u_m \rangle$$

with relations

$$\begin{aligned} u_1 \cdots u_{k_1} u_{k_2+1} \cdots u_{k_3} &= u_{k_1+1}^{a_{k_1+1}} \cdots u_{k_2}^{a_{k_2}} u_{k_3+1}^{a_{k_3+1}} \cdots u_{k_4}^{a_{k_4}} \\ u_1^{a_1} \cdots u_{k_1}^{a_{k_1}} u_{k_1+1}^{b_{k_1+1}} \cdots u_{k_2}^{b_{k_2}} u_{k_4+1}^{a_{k_4+1}} \cdots u_{k_5}^{a_{k_5}} &= u_{k_5+1} \cdots u_n, \end{aligned}$$

with $0 < k_1 \leq k_2 \leq k_3 \leq k_4 \leq k_5 < n \leq m$ and all $a_i, b_j \geq 1$ and (we agree that if $k_1 = k_2$, $k_2 = k_3$, $k_3 = k_4$ or $k_4 = k_5$ then the factors $u_{k_1+1}^{a_{k_1+1}} \cdots u_{k_2}^{a_{k_2}}$, $u_{k_1+1}^{b_{k_1+1}} \cdots u_{k_2}^{b_{k_2}}$, $u_{k_2+1} \cdots u_{k_3}$, $u_{k_3+1}^{a_{k_3+1}} \cdots u_{k_4}^{a_{k_4}}$, or $u_{k_4+1}^{a_{k_4+1}} \cdots u_{k_5}^{a_{k_5}}$ are the empty words). So, the two cases discussed in condition (3.d) of Theorem 3.1 correspond to $k_1 = k_2$ and $k_1 < k_2$, respectively.

As in the previous section, in order to compute the class group, we also may assume that $n = m$. Moreover, we may assume that $w_i \notin \{u_1, \dots, u_n\}$ for $i = 1, 2, 3, 4$, as otherwise S can be presented by a single relation and then the class group is given in Theorem 2.4. Under this restriction, in the next lemma, we describe the principal ideals as divisorial products of minimal prime ideals. Note that there are two possible types of minimal primes in S . First, there are

$$Q = (u_i, u_j),$$

where u_i and u_j each belong to the support of different sides of one of the defining relations and do not belong to the supports of the words in the other relation. To prove that Q is a prime ideal we may assume, by symmetry, that $u_i, u_j \in \text{supp}(w_1) \cup \text{supp}(w_2)$. Clearly, S/Q is then generated by the natural images of the elements $u_q, q \neq i, j$, subject to the unique relation $w_3 = w_4$. Since $u_i, u_j \notin \text{supp}(w_3) \cup \text{supp}(w_4)$, it is easily seen that $(S/Q) \setminus \{0\}$ is a multiplicatively closed set, as desired. Second, there are minimal primes of the form

$$Q = (u_i, u_j, u_k),$$

where u_i belongs to the support of a word in each of the two relations, u_j and u_k belong to the support of a word in a defining relation but on a different

side than u_i , and furthermore u_j and u_k are involved in different relations. In particular, $j \neq k$. Clearly, existence (and the number) of minimal primes of the latter type depends on the existence of strict inequalities $k_i < k_{i+1}$.

The formulas obtained in the following Lemma 3.2 should be interpreted in such a way that principal ideals Su_w and primes $P_{y,z}$ or $P_{t,v,x}$ are deleted if some index does not occur in the defining relations. So, for example P_{y,k_3+1} is not defined and hence ignored if $k_3 = k_4$.

Lemma 3.2 *Let*

$$S = \langle u_1, \dots, u_n \mid \begin{aligned} &u_1 \cdots u_{k_1} u_{k_2+1} \cdots u_{k_3} = u_{k_1+1}^{a_{k_1+1}} \cdots u_{k_2}^{a_{k_2}} u_{k_3+1}^{a_{k_3+1}} \cdots u_{k_4}^{a_{k_4}} \\ &u_1^{a_1} \cdots u_{k_1}^{a_{k_1}} u_{k_1+1}^{b_{k_1+1}} \cdots u_{k_2}^{b_{k_2}} u_{k_4+1}^{a_{k_4+1}} \cdots u_{k_5}^{a_{k_5}} = u_{k_5+1} \cdots u_n \end{aligned} \rangle,$$

with $0 < k_1 \leq k_2 \leq k_3 \leq k_4 \leq k_5 < n$ and all $a_i, b_j \geq 1$, be a normal monoid that cannot be presented with a single relation. Put $P_{y,z}$, the minimal prime ideal of S generated by $\{u_y, u_z\}$, $y \in \{k_2+1, \dots, k_3\}$, $z \in \{k_3+1, \dots, k_4\}$ or $y \in \{k_4+1, \dots, k_5\}$, $z \in \{k_5+1, \dots, n\}$ and put $P_{t,v,x}$, the minimal prime ideal of S that is generated by $\{u_t, u_v, u_x\}$, $t \in \{1, \dots, k_1\}$, $v \in \{k_1+1, \dots, k_2, k_3+1, \dots, k_4\}$, $x \in \{k_5+1, \dots, n\}$ or $t \in \{k_2+1, \dots, k_3\}$, $v \in \{k_1+1, \dots, k_2\}$, $x \in \{k_5+1, \dots, n\}$. Then

1. $Su_w = \left(\prod_{l=k_5+1}^n \left(\prod_{m=1}^{k_1} P_{m,w,l} \prod_{m=k_2+1}^{k_3} P_{m,w,l} \right) \right)^*$, for $w \in \{k_1+1, \dots, k_2\}$,
2. $Su_w = \left(\prod_{l=k_5+1}^n \left(\prod_{m=1}^{k_1} P_{m,w,l} \right) \right)^* * \left(\prod_{m=k_2+1}^{k_3} P_{m,w} \right)^*$, for $w \in \{k_3+1, \dots, k_4\}$,
3. $Su_w = \left(\prod_{l=k_5+1}^n P_{w,l} \right)^*$, for $w \in \{k_4+1, \dots, k_5\}$,
4. $Su_w = \left(\prod_{l=k_5+1}^n \left(\prod_{m=k_1+1}^{k_2} P_{w,m,l}^{a_m} \prod_{m=k_3+1}^{k_4} P_{w,m,l}^{a_m} \right) \right)^*$, for $w \in \{1, \dots, k_1\}$,
5. $Su_w = \left(\prod_{l=k_5+1}^n \left(\prod_{m=k_1+1}^{k_2} P_{w,m,l}^{a_m} \right) \right)^* * \left(\prod_{m=k_3+1}^{k_4} P_{w,m}^{a_m} \right)^*$, for $w \in \{k_2+1, \dots, k_3\}$,
6. $Su_w = \left(\prod_{l=1}^{k_1} \left(\prod_{m=k_1+1}^{k_2} P_{l,m,w}^{a_m} \prod_{m=k_3+1}^{k_4} P_{l,m,w}^{a_m} \right)^{a_l} \right)^* * \left(\prod_{m=k_1+1}^{k_2} \left(\prod_{l=1}^{k_1} P_{l,m,w} \prod_{l=k_2+1}^{k_3} P_{l,m,w} \right)^{b_m} \right)^* * \left(\prod_{l=k_4+1}^{k_5} P_{l,w}^{a_l} \right)^*$, for $w \in \{k_5+1, \dots, n\}$,

Proof. For $w \in \{1, \dots, n\}$, one notices that in the expressions for S_w , in the statement of the lemma, precisely all the minimal primes P occur that contain u_w . Using the defining relations one then easily verifies, as in the proof of Lemma 2.3, that the proposed formulae hold in the localizations S_P . Hence the result follows. ■

Our next aim is to describe the class group of S . Surprisingly, the proof is obtained by a reduction to the case considered in Theorem 2.4. The definitions of d_1 and d_2 in the following result should again be interpreted in the correct way when some $k_i = k_{i+1}$. We agree to ignore all a_i (respectively, b_j) for which u_i (respectively u_j) does not occur in the defining relations.

Theorem 3.3 *Let*

$$S = \langle u_1, \dots, u_n, \dots, u_m \mid \begin{aligned} u_1 \cdots u_{k_1} u_{k_2+1} \cdots u_{k_3} &= u_{k_1+1}^{a_{k_1+1}} \cdots u_{k_2}^{a_{k_2}} u_{k_3+1}^{a_{k_3+1}} \cdots u_{k_4}^{a_{k_4}} \\ u_1^{a_1} \cdots u_{k_1}^{a_{k_1}} u_{k_1+1}^{b_{k_1+1}} \cdots u_{k_2}^{b_{k_2}} u_{k_4+1}^{a_{k_4+1}} \cdots u_{k_5}^{a_{k_5}} &= u_{k_5+1} \cdots u_n \end{aligned} \rangle$$

(with $0 < k_1 \leq k_2 \leq k_3 \leq k_4 \leq k_5 < n \leq m$ and all $a_i, b_j \geq 1$) be a normal positive monoid that does not admit a presentation with a single defining relation. Let $Q = \{a_t a_v + b_v \mid t \in \{1, \dots, k_1\}, v \in \{k_1 + 1, \dots, k_2\}\} \cup \{a_t a_v \mid t \in \{1, \dots, k_1\}, v \in \{k_3 + 1, \dots, k_4\}\} \cup \{a_y \mid y \in \{k_4 + 1, \dots, k_5\}\}$. Then

$$\text{cl}(K[S]) \cong \text{cl}(S) \cong \mathbb{Z}^f \times (\mathbb{Z}_{d_1})^{k_1+k_3-k_2-1} \times (\mathbb{Z}_{d_2})^{n-k_5-1},$$

where

$$\begin{aligned} f &= (k_3 - k_2)(k_4 - k_3) + (k_5 - k_4)(n - k_5) + k_1(k_4 - k_3 + k_2 - k_1)(n - k_5) \\ &\quad + (k_3 - k_2)(k_2 - k_1)(n - k_5) - (n - 2), \end{aligned}$$

with

$$d_1 = \text{gcd}(a_{k_1+1}, \dots, a_{k_2}, a_{k_3+1}, \dots, a_{k_4})$$

and

$$d_2 = \begin{cases} \text{gcd}(a_1 d_1, \dots, a_{k_1} d_1, b_{k_1+1}, \dots, b_{k_2}, a_{k_4+1}, \dots, a_{k_5}) & \text{if } k_2 < k_3 \\ \text{gcd}(q \mid q \in Q) & \text{if } k_2 = k_3 \end{cases}.$$

Proof. As mentioned earlier, without loss of generality we may assume that $n = m$. It is shown in the proof of Theorem 3.1 that $SS^{-1} \cong \text{Fa}_{n-2}$, the free abelian group of rank $n - 2$. Because $U(S) = \{1\}$, we get that $P(S)$ and SS^{-1} are isomorphic, and thus they have the same torsion free rank. Since the torsion free rank of $\text{cl}(S)$ is the difference of the torsion free rank of $D(S)$ and the torsion free rank of $P(S)$, to establish the description of the torsion free part of $\text{cl}(S)$, we only need to show that there are $(k_3 - k_2)(k_4 - k_3) + (k_5 - k_4)(n - k_5) + k_1(k_4 - k_3 + k_2 - k_1)(n - k_5) + (k_3 - k_2)(k_2 - k_1)(n - k_5)$ minimal primes in S . But this easily follows from the description of the minimal primes given Lemma 3.2.

As in the proof of Theorem 2.4, we consider $\text{cl}(S)$ as a finitely generated \mathbb{Z} -module, so that its presentation is determined by an integer matrix M of size $r \times n$, where r is the number of minimal primes in S , hence the basis of $D(S)$. Therefore, the rows are indexed by all triples (t, v, x) and all pairs (y, z) , as described in Lemma 3.2. We agree on the following ordering of the set of rows of M : all triples (t, v, x) are ordered lexicographically, so are all the pairs (y, z) and $(t, v, x) < (y, z)$ for every t, v, x, y, z . The columns are indexed by

$1, 2, \dots, n$, where the i -th column corresponds to the generator Su_i , written as a vector in terms of the minimal primes of S , as in Lemma 3.2.

We consider the block decomposition of M determined by the following partitions of the sets C and R of columns and rows:

$$C = D_1 \cup D_2,$$

where $D_1 = \{1, \dots, k_5\}$ and $D_2 = \{k_5 + 1, \dots, n\}$. Notice that $|D_2| \geq 2$ because S does not admit a presentation with one defining relation. Let $R_0 = \{(y, z) \mid y \in \{k_2 + 1, \dots, k_3\}, z \in \{k_3 + 1, \dots, k_4\}\}$, $R_y = \{(y, z) \mid z \in \{k_5 + 1, \dots, n\}\}$ for $y \in \{k_4 + 1, \dots, k_5\}$. For every triple (t, v, x) we also define $R_{t,v} = \{(t, v, x) \mid x \in \{k_5 + 1, \dots, n\}\}$. Then

$$R = \bigcup R_{t,v} \cup R_0 \cup \bigcup_{y=k_2+1}^{k_3} R_y,$$

where the first union runs over all pairs (t, v) such that the set R of rows contains a triple of the form (t, v, x) .

Consider any of the block submatrices $M_{R_{t,v}, C}$ or $M_{R_y, C}$, with $R_{t,v}, R_y$ as above. From Lemma 3.2 it follows that, ignoring the zero columns of this submatrix, it has the form

$$\begin{pmatrix} a & b & d & 0 & \cdots & 0 \\ a & b & 0 & d & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ a & b & 0 & \cdots & 0 & d \end{pmatrix},$$

for some a, b such that either $a = 1$ or $b = 1$ and for some d . Here the columns of the scalar matrix determined by d are indexed by D_2 . So, subtracting the first row in each such block ($M_{R_{t,v}, C}$ or $M_{R_y, C}$) from all the remaining rows in this block and next subtracting the last $n - (k_5 + 1)$ columns of the entire matrix from column $k_5 + 1$, we get a matrix M' such that each block M'_{X, D_1} , for $X = R_y$ or $X = R_{t,v}$, has only the first row nonzero and $M'_{R, D_2} = M_{R, D_2}$. Moreover $M'_{R_0, C} = M_{R_0, C}$. Therefore, the nonzero entries of the last column of M' are the only nonzero entries in their respective rows. Denote by Y the set of all such rows of M' . Then these nonzero entries (in the last column of M'), and with our convention as explained before the theorem, are:

$$\begin{array}{llll} a_t a_v + b_v & \text{for} & t \in \{1, \dots, k_1\}, v \in \{k_1 + 1, \dots, k_2\} & \text{if } k_1 \neq k_2 \\ a_t a_v & \text{for} & t \in \{1, \dots, k_1\}, v \in \{k_3 + 1, \dots, k_4\} & \text{if } k_3 \neq k_4 \\ b_t & \text{for} & t \in \{k_1 + 1, \dots, k_2\} & \text{if } k_1 \neq k_2, k_2 \neq k_3 \\ a_y & \text{for} & y \in \{k_4 + 1, \dots, k_5\} & \text{if } k_4 \neq k_5. \end{array}$$

Notice that the greatest common divisor of the specified set of elements is equal to d_2 , as defined in the statement of the theorem. Thus, row elimination within the block $M'_{Y, C}$ allows us to produce a row of the form $(0, \dots, 0, d_2)$ and replace all other rows of $M'_{Y, C}$ by zero rows. The same argument can be applied to the

nonzero entries in the subsequent columns: $n-1, n-2, \dots, k_5+2$. This leads to a matrix M'' (of the same size as the original matrix M) with $n-k_5-1$ rows of the form $(0, \dots, d_2, 0, \dots, 0)$, with d_2 in positions k_5+2, \dots, n , with no other nonzero entries in their respective columns. So, it remains to find the normal form of the matrix N obtained by deleting in M'' the last $n-k_5-1$ columns and the rows that contain the nonzero entries in these columns. It is easy to see that the last column of N is a \mathbb{Z} -combination of the remaining columns. Namely, we have $C_{k_5+1} = a_1 C_1 + \dots + a_{k_1} C_{k_1} + b_{k_1+1} C_{k_1+1} + \dots + b_{k_2} C_{k_2} + a_{k_4+1} C_{k_4+1} + \dots + a_{k_5} C_{k_5}$. Hence by column operations we can make this column zero. Then, deleting this column, we get a matrix with k_4 columns that is of the form $\begin{pmatrix} N' & 0 \\ 0 & I \end{pmatrix}$ for a matrix N' and the identity $(k_5-k_4) \times (k_5-k_4)$ -matrix I . It is easy to see that N' corresponds to the monoid T with the presentation $u_1 \cdots u_{k_1} u_{k_2+1} \cdots u_{k_3} = u_{k_1+1}^{a_{k_1+1}} \cdots u_{k_2}^{a_{k_2}} u_{k_3+1}^{a_{k_3+1}} \cdots u_{k_4}^{a_{k_4}}$ and with the generating set u_1, \dots, u_{k_4} . Hence, by Theorem 2.4, $\text{cl}(T) = \mathbb{Z}^e \times \mathbb{Z}_{d_1}^{k_3-k_2+k_1-1}$, where $e = (k_1+k_3-k_2)(k_2-k_1+k_4-k_3) - (k_4-1)$. Therefore, the normal form of M has $k_3-k_2+k_1-1$ copies of d_1 and $n-k_5-1$ copies of d_2 and a certain number of entries equal to 1. By the comment at the beginning of the proof, it must have f zero rows. Hence, the result follows. ■

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References

- [1] W. Bruns and J. Gubeladze, Semigroup algebras and discrete geometry, Séminaire & Congrès 6 (2002), Soc. Math. France, Paris, 43–127.
- [2] W. Bruns and J. Herzog, Cohen - Macaulay Rings, Cambridge Univ. Press, rev. ed., Cambridge, 1998.
- [3] S.T. Chapman, P.A. Garcia-Sánchez, D. Llena and J.C. Rosales, Presentations of finitely generated cancellative abelian monoids and nonnegative solutions of systems of linear equations, Discrete Appl. Math. 154 (2006), 1947–1959.
- [4] L.G. Chouinard II., Krull semigroups and divisor class groups, Canad. J. Math. 23 (1981), 1459–1468.
- [5] P. Dueck, S. Hosten and B. Sturmfels, Normal toric ideals of low codimension, arXiv:0801.3826, preprint.

- [6] P. Etingof, T. Schedler and A. Soloviev, Set-theoretical solutions to the quantum Yang-Baxter equation, *Duke Math. J.* 100 (1999), 169–209.
- [7] R. Fossum, *The Divisor Class Group of a Krull Domain*, Springer-Verlag, New York, 1973.
- [8] T. Gateva-Ivanova and M. Van den Bergh, Semigroups of I -type, *J. Algebra* 206 (1998), 97–112.
- [9] R. Gilmer, *Commutative Semigroup Rings*, Univ. Chicago Press, Chicago, 1984.
- [10] I. Goffa, E. Jespers, Monoids of IG-type and maximal orders, *J. Algebra* 308 (2007), 44–62.
- [11] I. Goffa, E. Jespers and J. Okniński, Primes of height one and a class of Noetherian finitely presented algebras, *Internat. J. Algebra Comput.* 17 (2007), 1465–1491.
- [12] E. Jespers and J. Okniński, *Noetherian Semigroup Algebras, Algebra and Applications*, Springer, 2007.
- [13] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra*, Graduate Texts in Mathematics 227, Springer, 2005.
- [14] H. Ohsugi and T. Hibi, Toric ideals arising from contingency tables, in: *Commutative algebra and combinatorics. Part I: Computational algebra and combinatorics of toric ideals. Part II: Topics in commutative algebra and combinatorics. Ramanujan Mathematical Society Lecture Notes Series* 4, pp. 91–115, 2007.
- [15] J.C. Rosales and P.A. Garcia-Sánchez, On normal affine semigroups, *Linear Algebra Appl.* 286 (1999), 175–186.
- [16] A. Simis and R.H. Villareal, Constraints for the normality of monomial subrings and birationality, *Proc AMS* 131 (2002), 2043–2048.
- [17] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, Univ. Lect. Ser. 8, Amer. Math. Soc., 1996.
- [18] R.H. Villareal, *Monomial Algebras*, Marcel Dekker, 2001.

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